THE FUNDAMENTAL GROUP

FUNDAMENTAL THEORE OF ALGEBRA :—

Every polynomial equation with real or complex co-efficients has a root.

JORDAN CURVE THEOREM :—

Every simple closed curve C in the plane separates the plane into two components, of which C is the common boundary.

HOMOTOPY

If f and f' are continuous maps of the space X into the space Y, we say that f is homotopic to f' if there is a continuous map $H : X \times I \to Y$ such that

$H(x, 0) = f(x)$ and $H(x, 1) = f'(x)$, for each x, $I = [0, 1]$. —— (I)

The map H is called a homotopy between f and f'. we write $f \simeq f'$.

If f is homotopic to f', we say that f is

If $f \simeq f'$ and f' is a constant map, we say that f is null homotopic.

- We think of a homotopy as a continuous one-parameter family of maps from X to Y. If the parameter t representing the time, then the homotopy H represents a continuous "deforming" of the map f to the map f', as t goes from 0 to 1.
- Let us focus on the particular case when $X = [a, b]$ then if $f : [a, b] \to Y$ is a continuous function then f is called a path in Y, $f(a)$ is said to be the initial point and $f(b)$ is said to be the final point.

Path homotopic

Let $f : I \to X$ and $f' : I \to X$, where $I = [0, 1]$ be two paths in X; are said to be path homotopic if they have the same initial point $f(0) = f'(0) = x_0$ and the same final point $f(1) = f'(1) = x_1$ and if there is a continuous map $H : I \times I \to X$ such that

$H(s, 0) = f(s)$ and $H(s, 1) = f'(s)$

$H(0, t) = x_0$ and $H(1, t) = x_1$

for each $s \in I$ and each $t \in I$. We call H a path homotopy between f and f', written as $f \simeq_p f'$. —— (II)

(ii) $\Rightarrow$ The end points of the path remains fixed during the deformation.

Lemma

The relation $\simeq$ and $\simeq_p$ are equivalence relation

proof:-

i) Given f, it is trivial that $f \simeq f'$, we show that $f' \simeq f$. There map, $H(x,t) = f(x)$ is the required homotopy.

$H(x,0) = f(x)$, $H(x,1) = f(x)$.

~~So, f & f is homotopic~~

So, f is homotopic to f. ———— (reflexive)

ii) Given $f \simeq f'$, we show that $f' \simeq f$. Let H be a homotopy between f & f'. Then $G(x,t) = H(x, 1-t)$ is a homotopy between f' and f.

$$G(x,0) = H(x,1) = f'(x)$$
$$G(x,1) = H(x,0) = f(x).$$

Then $f' \simeq f$. ———— (symmetric)

iii) Suppose that $f \simeq f'$ and $f' \simeq f''$. we show that $f \simeq f''$. Let H be a homotopy between f & f' and H' be a homotopy between f' & f''. Define $G : X \times I \longrightarrow Y$ be the eqn

$$G(x,t) = \begin{cases} H(x, 2t) & \text{for } t \in [0, 1/2] \\ H'(x, 2t-1) & \text{for } t \in [1/2, 1] \end{cases}$$

$$\boxed{\begin{aligned} H(x, 2\times 1/2) &= f'(x) \\ H'(x, 2\times 1/2 - 1) &= f'(x) \end{aligned}}$$

$G(x,t)$ well defined, since $t = 1/2$, $H(x, 2t) = f'(x)$
$H'(x, 2t-1) = f'(x)$.

Because G is continuous on the two closed subsets $X \times [0, 1/2]$ and $X \times [1/2, 1]$ of $X \times I$, it is continuous on all of $X \times I$ by pasting lemma.

$G(x,0) = H(x,0) = f(x)$, $G(x,1) = H'(x, 1) = f''(x)$.

Therefore $f \cong f''$. ———— (transitive)

Scanned with CamScanner

* Let f is a path, $t \in [0,1]$, $f(0) = x_0$, $f(1) = x_1$.

i) $H(x,t) = f(x)$.

$H(0,t) = f(0) = x_0$, $\quad H(1,t) = f(1) = x_1$.

$f \simeq_p f$.

ii) $G(x,t) = H(x, 1-t)$. $\qquad H(0,t) = x_0$, $H(1,t) = x_1$.

$G(0,t) = x_0$, $\quad G(1,t) = x_1$ $\qquad f \simeq_p f' \Rightarrow f' \simeq_p f$.

iii) $G(x,t) = \begin{cases} H(x, 2t) & t \in [0, \tfrac{1}{2}] \\ H'(x, 2t-1) & t \in [\tfrac{1}{2}, 1]. \end{cases}$

$G(0,t) = \begin{cases} H(0, 2t) \\ H'(0, 2t-1) \end{cases} > x_0$ $\qquad f''(0) = f'(0) = f(0) = x_0$, $\quad f''(1) = f'(1) = f(1) = x_1$

$G(1,t) = \begin{cases} H(1, 2t) \\ H'(1, 2t-1) \end{cases} > x_1$

$f \simeq_p f'$, $f' \simeq_p f'' \Rightarrow f \simeq_p f''$.

Example

Let f and g be any two maps of a space X into $\mathbb{R}^2$. It is easy to see that f and g are homotopic; the map

$$F(x,t) = (1-t) f(x) + t\, g(x)$$

is a homotopy between them. It is called a straight-line homotopy because it moves the point $f(x)$ to the point $g(x)$ along the st. line segment joining them.

[• If f is a path, we shall denote its path homotopy equivalence class by $[f]$.]

• If f and g are from x_0 to x_1, then F will be a path homotopy.

1) Let $f: I \to \mathbb{R}^2$, $g: I \to \mathbb{R}^2$ be paths with $f(0) = g(0)$ and $f(1) = g(1)$.

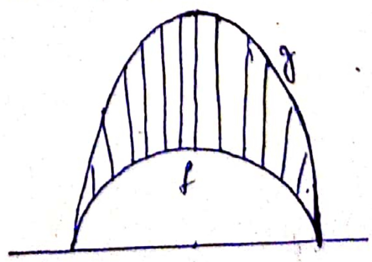
Define $H: I \times I \to \mathbb{R}^2$ by $H(x,t) = (1-t) f(x) + t\, g(x)$.

2) Let $f: I \to \mathbb{R}^2$ be given by

$f(s) = (\cos \pi s, \sin \pi s)$

$g: I \to \mathbb{R}^2$ be given by

$g(t) = (\cos \pi t, 3 \sin \pi t)$.



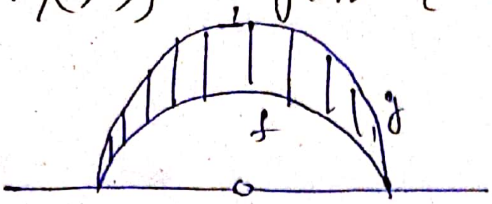
$f(0) = (1,0) = g(0)$

$f(1) = (-1, 0) = g(1)$. $\quad f$ & g are path homotopy.

Scanned with CamScanner

3. $f : I \to \mathbb{R}^2 \setminus \{(0,0)\}$ $f(s) = (\cos \pi s, \sin \pi s)$

$g : I \to \mathbb{R}^2 \setminus \{(0,0)\}$ $g(t) = (\cos \pi t, 3 \sin \pi t)$

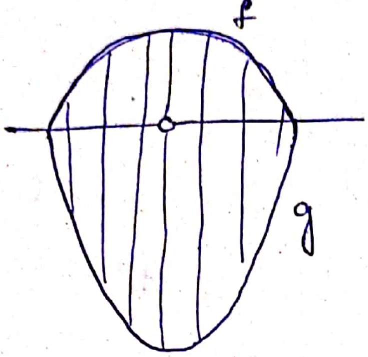


path homotopy.

4) $f(s) = (\cos \pi s, \sin \pi s)$ $g(t) = (\cos \pi t, -3 \sin \pi t)$

$f : I \to \mathbb{R}^2 \setminus \{(0,0)\}$

$g : I \to \mathbb{R}^2 \setminus \{(0,0)\}$.



not path homotopy.

$f \not\simeq_p g$.

Path Product

If f is a path in X from x_0 to x_1, $f : I \to X$ s.t. $f(0) = x_0$, $f(1) = x_1$ and $g : I \to X$ is a path in X (with $g(0) = x_1$ and $g(1) = x_2$) from x_1 to x_2, we define the product $f * g$ of f & g to be the path h given by the eqn

$$h(s) = \begin{cases} f(2s) & \text{for } s \in [0, \tfrac{1}{2}] \\ g(2s-1) & \text{for } s \in [\tfrac{1}{2}, 1] \end{cases}$$

The function h is well-defined and continuous, by ~~past~~ pasting lemma; it is a path in X from x_0 to x_2.

- The product operation on paths induces a well-defined operation on path-homotopy classes, defined by the eqn,

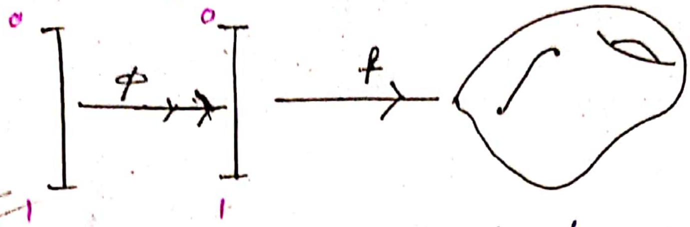
$$[f] * [g] = [f * g].$$

Show that $f \cdot g \simeq_p f' \cdot g'$

Reparametrisation

We define reparametrization of a path f to be a composition $f \circ \phi$, where $\phi : I \to I$ is a continuous map s.t. $\phi(0) = 0$ and $\phi(1) = 1$.

Reparametrizing a path preserves its homotopy class, i.e. $[f \circ \phi] = [f]$.



Let us consider the path homotopy $f \circ \phi_t$ where $\phi_t(s) = (1-t)\,\phi(s) + t \cdot s$

$\phi_0(s) = \phi(s) \Rightarrow f \circ \phi_0 = f \circ \phi$.

$\phi_1(s) = s \Rightarrow f \circ \phi_1 = f$.

$\cancel{H(0,s) = f \circ \phi}$ $\quad H(s,0) = f \circ \phi$.

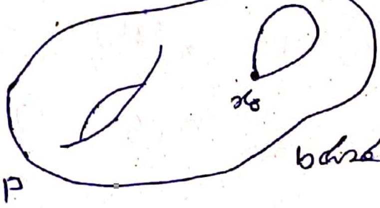
$\cancel{H(1,s) = f}$. $\quad H(s,1) = f$.

$\therefore [f \circ \phi] = [f]$.

Loop $\quad f : I \xrightarrow{\text{Cont.}} X$

$f(0) = f(1)$

X be a space, $x_0 \in X$. A path f in X that begins & end at x_0, is called loop



based at x_0.

THE FUNDAMENTAL GROUP

Let $f : I \to X$ be a path s.t. $f(0) = f(1) = x_0$. Such a path is said to be a loop in X based at x_0.

The set of path homotopy classes of loops based at x_0, with the operation $*$, is called fundamental group of X relative to the base point x_0.

The set $\{[f] : f : I \xrightarrow{\text{Cont.}} X,\ f(0) = f(1) = x_0\}$ is denoted as $\pi_1(X, x_0)$. In this set let us consider the product $[f] \cdot [g] = [f \cdot g]$, $g \in \pi_1(X, x_0)$. Then $(\pi_1(X, x_0), \cdot)$ is a group or, fundamental group of X.

proof :—

1. Associativity :—

To show that $[f]\left([g] \cdot [h]\right) = \left([f] \cdot [g]\right)[h]$.

where $[f], [g], [h] \in \pi_1(X, x_0)$.

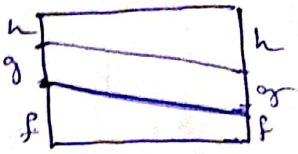
$[f]\cdot([g]\cdot[h]) = ([f]\cdot[g])\cdot[h]$

$\Rightarrow [f]\cdot[g\cdot h] = [f\cdot g]\cdot[h]$

$\Rightarrow [f\cdot(gh)] = [(f\cdot g)\cdot h]$.

i.e. we have to show that $f\cdot(g\cdot h) \simeq (f\cdot g)\cdot h$.

This will be done if we show that $f\cdot(g\cdot h)$ is a reparametrisation of $(f\cdot g)\cdot h$.



Let us define a function $\phi : I \to I$ s.t.

$$\phi(t) = \begin{cases} t/2, & 0 \le t \le 1/2 \\ t - 1/4, & 1/2 \le t \le 3/4 \\ 2t - 1, & 3/4 \le t \le 1. \end{cases}$$

Now, $(f\cdot(g\cdot h))(t) = f(2t), \quad 0 \le t \le 1/2$

$= (g\cdot h)(2t-1), \quad 1/2 \le t \le 1$

$\Rightarrow f\cdot(g\cdot h)(t) = f(2t), \quad 0 \le t \le 1/2$

$= g(4t-2), \quad 1/2 \le t \le 3/4$

$= h(4t-3), \quad 3/4 \le t \le 1$

Analogously, $((f\cdot g)\cdot h)(s) = (f\cdot g)(2s), \quad 0 \le s \le 1/2$

$= h(2s-1), \quad 1/2 \le s \le 1$

$((f\cdot g)\cdot h)(s) = f(4s), \quad 0 \le s \le 1/4$

$= g(4s-1), \quad 1/4 \le s \le 1/2$

$= h(2s-1), \quad 1/2 \le s \le 1.$

Now, $([(f\cdot g)\cdot h]\circ\phi)(t) = f(2t), \quad 0 \le t \le 1/2$

$= g(4t-2), \quad 1/2 \le t \le 3/4$

$= h(4t-3), \quad 3/4 \le t \le 1.$

$\therefore ((f\cdot g)\cdot h)\circ\phi = (f\cdot(g\cdot h))$.

Thus $[(f\cdot g)\cdot h] = [f\cdot(g\cdot h)]$.

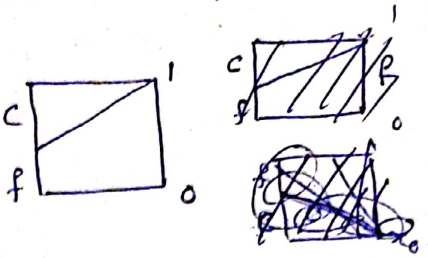
Scanned with CamScanner

2. Identity:

Consider $c : I \to X$ with $c(s) = x_0, \forall s \in I$.

$f.c$ is a reparametrisation of f via the map $\phi : I \to I$ defined by

$$\phi(t) = 2t, \quad 0 \le t \le 1/2$$
$$= 1, \quad 1/2 \le t \le 1.$$



Therefore, $(f.c)(t) = f(2t), \quad 0 \le t \le 1/2$
$$= x_0, \quad 1/2 \le t \le 1.$$

$$(f \circ \phi)(t) = f(2t), \quad 0 \le t \le 1/2$$
$$= f(1) = x_0, \quad 1/2 \le t \le 1.$$

$(f \circ \phi)(t) = (f.c)(t). \Rightarrow f \simeq f.c$

$\therefore [f] = [f.c] = [f].[c]$

Similarly $[f] = [c.f] = [c].[f]$.

Hence $[c] \in \pi_1(X, x_0)$ is the identity.

3. Inverse: Consider $[f] \in \pi_1(X, x_0)$ and $\bar{f}(s) = f(1-s)$.

We shall show that $f.\bar{f} \simeq c \simeq \bar{f}.f$.

Consider, $H(s,t) = f(2s), \quad 0 \le s \le \frac{1-t}{2}$
$$= f(1-t), \quad \frac{1-t}{2} \le s \le \frac{1+t}{2}$$
$$= f(2-2s), \quad \frac{1+t}{2} \le s \le 1$$

$(f.\bar{f})(t) = f(2t) \quad 0 \le s \le 1/2$
$= \bar{f}(2t-1) = \cancel{f(2(1-t)-1)} = \cancel{f}(\cdot f(1-2s+1) = f(2-2s)$

$H(s,0) = f(2s) \quad 0 \le s \le 1/2$
$= f(2-2s), \quad 1/2 \le s \le 1.$

$H(s,1) = f(0) = x_0 = c(t).$

H is a homotopy between $f\bar{f} \simeq c$.

i.e. $[f][\bar{f}] = [f.\bar{f}] = [c]$.

Therefore, $\pi_1(X, x_0)$ is a group and it is call the fundamental group of X at x_0.

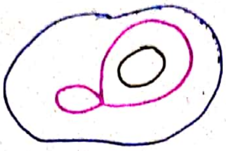
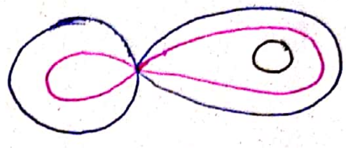
Ex $X = \mathbb{R}^n$, $x_0 \in \mathbb{R}^n$

1. st. line homotopy, $H(x,t) = (1-t)x_0 + t x_0 = c(t)$

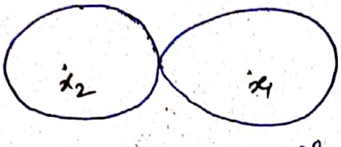
$$\pi_1(\mathbb{R}^n, x_0) = \{[c]\}.$$

2. Let X be a convex subset of $\mathbb{R}^n$

$$\pi_1(\mathbb{R}^2, x_0) = \{[c]\}.$$

4 elements

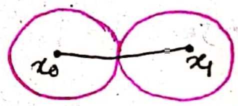


$\pi_1(X, x_1) \cong \pi_1(X, x_2)$

Contain same no. of element.

Proposition

If x_0, x_1 are in the same path component of X, then $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic.



proof :-

Let $x_0, x_1 \in X$. Since x_0, x_1 are in the same path component, there is a path connecting x_0 and x_1. Let $h: I \to X$ be a path with $h(0) = x_0$ and $h(1) = x_1$. Let $\bar{h}: I \to X$ be given by

$\bar{h}(s) = h(1-s)$, $s \in I = [0,1]$.

Define $\hat{h}: \pi_1(X, x_0) \to \pi_1(X, x_1)$ by

$$\hat{h}([f]) = [h \cdot f \cdot \bar{h}].$$

i) $\hat{h}$ is well defined : If $[f] = [g] \in \pi_1(X, x_0)$ and f_t be a homotopy between f & g then $h \cdot f_t \cdot \bar{h}$ is a homotopy between $h \cdot f \cdot \bar{h}$ and $h \cdot g \cdot \bar{h}$, i.e. $[h \cdot f \cdot \bar{h}] = [h \cdot g \cdot \bar{h}]$.

ii) $\hat{h}$ is a homomorphism :-

$$\begin{aligned} \hat{h}([f]\cdot[g]) &= [h \cdot f \cdot g \cdot \bar{h}] \\ &= [h \cdot f \cdot \bar{h} h \cdot g \cdot \bar{h}] \\ &= [h \cdot f \cdot \bar{h}][h \cdot g \cdot \bar{h}] \\ &= \hat{h}([f])\, \hat{h}([g]). \end{aligned}$$

(iii) $\hat{h}$ is bijective :-

Let $\hat{h}([f]) = \hat{h}([g])$

$\Rightarrow [h \cdot f \cdot \bar{h}] = [h \cdot g \cdot \bar{h}]$

$\Rightarrow [\bar{h}] \cdot [f] \cdot [\bar{h}] = [\bar{h}]\,[g]\,[\bar{h}]$

$\Rightarrow [f] = [g] \longrightarrow \hat{h}$ is one to one.

Now, let, $[\sigma] \in \pi_1(X, x_0)$.

$[\bar{h}, \sigma, h] \in \pi(X, x_1)$

$\Rightarrow \hat{h}([\bar{h} \cdot \sigma \cdot h]) = [h\bar{h} \cdot \sigma \cdot h\bar{h}] = [\sigma]$.

If $[f] = [\bar{h} \cdot \sigma \cdot h]$.

then $\hat{h}([f]) = \sigma$.

So, $\hat{h}$ is surjective.

So, $\hat{h}$ is an isomorphism between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$.

Definition

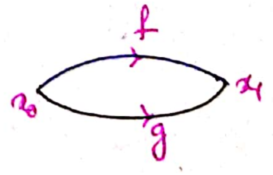
X is said to be simply ~~con~~ connected if it is path connected and $\pi_1(X)$ is trivial.

Proposition

X is simply connected iff there is a unique homotopy class of paths connecting any two points in space.

In a simply connected space X, any two paths having the same initial and final points are path homotopy.

proof:-



Let f and g be two paths from x_0 to x_1. Then $f \circ \bar{g}$ is defined and is a loop on X based at x_0.

Since X is simply connected,

$f \circ \bar{g} \sim_p e_{x_0}$.

Applying the groupoid properties,

$$[(f\circ\bar{g})\circ g] = [e_{x_0}\circ g] = [g]$$

$$[(f\circ\bar{g})\circ g] = [f\circ(\bar{g}\circ g)] = [f\circ e_{x_0}] = [f].$$

Then $[f] = [g]$.

So, f & g are path homotopy.

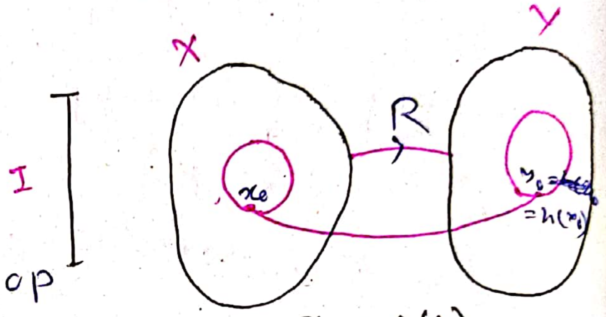
* *

Let X, Y be topological spaces, $x_0 \in X$, $y_0 \in Y$ and $h: X \to Y$ be continuous with $h(x_0) = y_0$.

We shall denote this by

$h:(X, x_0) \longrightarrow (Y, y_0)$.

If $f: I \to X$ is a loop in X based at x_0, then $h\circ f$ is a loop in Y based at y_0.



$f(0) = x_0 = f(1)$

base at x_0.

$h\circ f(0) = y_0$

$h\circ f(1) = y_0$.

Definition

Let $h:(X, x_0) \longrightarrow (Y, y_0)$ be continuous. Define $h_*: \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ by $h_*([f]) = [h\circ f]$.

Show that $h_*([f]) = [h\circ f]$ is a homomorphism.

i) well defined.

$[f] = [g] \in (X, x_0)$

$\Rightarrow [h\circ f] = [h\circ g]$

ii) $h_*([f]\cdot[g]) = h_*([f\cdot g])$

$= [h\circ(f\cdot g)]$

$[h\circ f]\cdot[h\circ g] = [(h\circ f)\cdot(h\circ g)]$

$= h\circ(f\cdot g)$

This is called the induced 'homomorphism' by h relativity to x_0.

* Two space are homeomorphism
= their fundamental groups are isomorphic.

Proposition

1. If $h:(X, x_0) \longrightarrow (Y, y_0)$, $g:(Y, y_0) \longrightarrow (Z, z_0)$ be continuous, then $(g \circ h)_* = g_* \circ h_*$

2. If $I_* : (X, x_0) \to (X, x_0)$ then $(I_*)_* = I_{\pi_1(X, x_0)}$

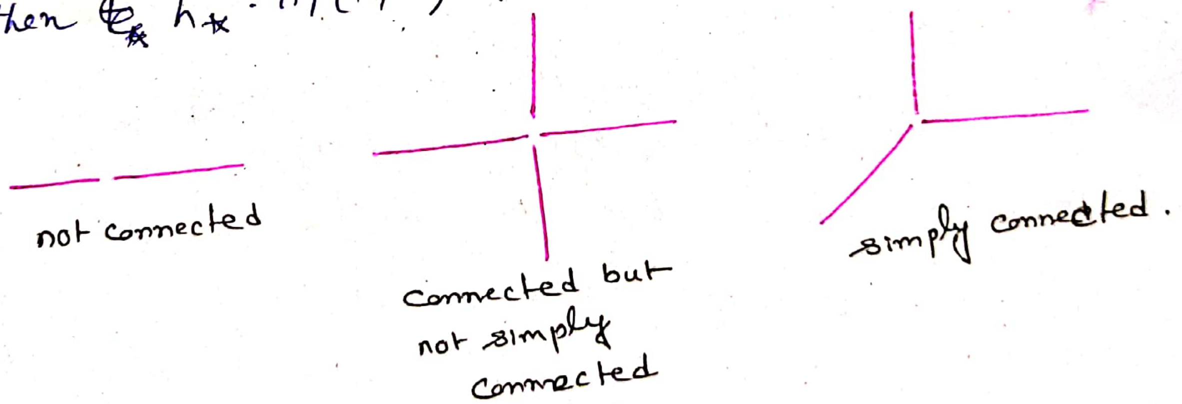
$$\phi(I_x)([f]) = [I_x \circ f]$$
$$= [f]$$
$$= I_{\pi_1(X, x_0)}([f]).$$

$$(I_x)_* \approx I_{\pi_1(X, x_0)}$$ ~~homeomorphis~~

* $h_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$
* $g_* : \pi_1(Y, y_0) - \pi_1(Z, z_0)$

Proposition

If $h:(X, x_0) \longrightarrow (Y, y_0)$ is a homeomorphism then $h_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ is a group isomorphism



Categories and functors

A category $\mathcal{D}$ consists of

1. a collection of objects $Ob(\mathcal{D})$
2. sets of morphisms $Mor(X,Y)$ for each $X, Y \in Ob(\mathcal{D})$ with a distinguished identity morphism $I_X \in Mor(X,X)$
3. a composition of morphisms '$\circ$' : $Mor(X,Y) \times Mor(Y,Z) \to Mor(X,Z)$ for each $X, Y, Z \in Ob(\mathcal{D})$ with the properties,

$$f \circ I = I \circ f = f \text{ and } (f \circ g) \circ h = f \circ (g \circ h).$$

Ex

1. $Ob(\mathcal{D}) = \{G : G \text{ is a group}\}$

   $Mor(G,H) = \{f : G \to H \mid f \text{ is a group homomorphism}\}$

2. $Ob(\mathcal{D}) = \{(X, x_0) \mid X \text{ is a topological space}, x_0 \text{ is a base point}\}$

   $Mor((X,x_0),(Y,y_0)) = \{f : X \to Y \text{ with } f(x_0) = y_0, \text{ where } f \text{ is continuous}\}$.

* *

A covarient functors F from a category $\mathcal{C}$ to $\mathcal{D}$ assigns to each $x \in Ob(\mathcal{C})$ and $F(x) \in Ob(\mathcal{D})$ and to each morphism $f \in Mor(X,Y)$ a $F(f)$ such that

$$F(I_X) = I_{F(X)} \text{ and } F(f \circ g) = F(f) \circ F(g).$$

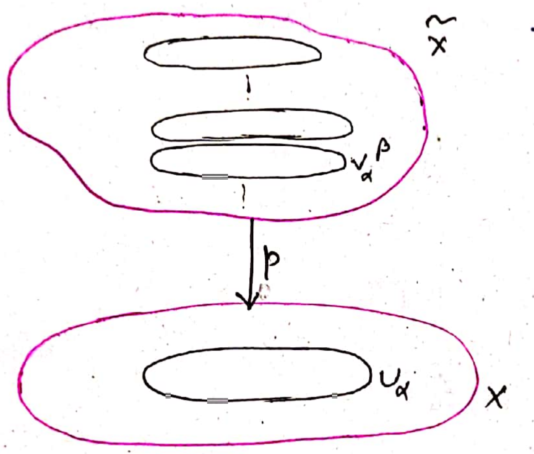
Ex

π_1 is a covarient functor from category of pointed topological spaces to the category of groups.

Scanned with CamScanner

Covering Space

Let $p : \tilde{X} \to X$ be a continuous and surjective map. If there exists an open cover $\{U_\alpha\}_{\alpha \in \Lambda}$ of X such that for each $\alpha \in \Lambda$, there exists $p^{-1}(U_\alpha) = \bigcup_{\beta \in \Gamma_\alpha} V_\alpha^\beta$, where $V_\alpha^\beta \cap V_\alpha^{\beta'} = \phi$ for any $\beta, \beta' \in \Gamma_\alpha$ with $\beta \neq \beta'$ and $p|_{V_\alpha^\beta} : V_\alpha^\beta \to U_\alpha$ is a homeomorphism for each $\beta \in \Gamma_\alpha$ where V_α^β is open in $\tilde{X}$.



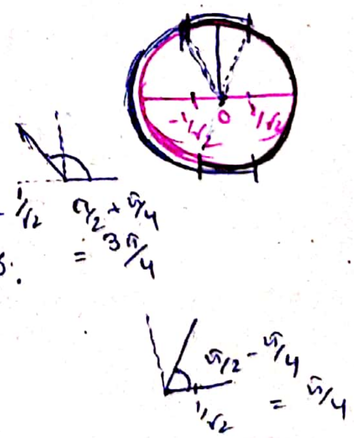
Then p is said to be covering map and $\tilde{X}$ is said to be a covering space of X.

<u>Ex</u> 1. $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

Let $p : \mathbb{R} \to S^1$ be given by
$$p(s) = (\cos 2\pi s, \sin 2\pi s).$$

Let $U_1 = \{(x,y) \in S^1 : x > -\sqrt{2}/2 = -1/\sqrt{2}\}$

$U_2 = \{(x,y) \in S^1 : x < 1/\sqrt{2}\}$



$\Rightarrow p^{-1}(U_1) = \bigcup_{n \in \mathbb{Z}} V_1^n$ ~~where~~

where, $V_1^n = (n - 3/8, n + 3/8)$

$p : V_1^n \to U_1$ is a homeomorphism, $V_1^n \cap V_1^{n'} = \phi$.

Similarly, $p^{-1}(U_2) = \bigcup_{n \in \mathbb{Z}} V_2^n$

where, $V_2^n = (n - 1/8, n + 1/8)$

$p : V_2^n \to U_2$ is a homeomorphism, $V_2^n \cap V_2^{n'} = \phi$.

THEOREM

Let $p:\tilde{X}\to X$ be a covering map and $X_0\subseteq X$. Let $\tilde{X}_0=p^{-1}(X_0)$, then $p_0:\tilde{X}_0\longrightarrow X_0$ obtained by restricting p on $\tilde{X}_0$ is a covering map.

proof:-

~~Obio~~ Obviously, p_0 is a continuous map, (restricting the domain or range of a continuous function gives a continuous function). Also, p_0 is surjective. Now let us consider an open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of X with the properties in the definition of a covering map. Then for each $\alpha\in\Lambda$, $p^{-1}(U_\alpha)=\bigcup_{\beta\in\Gamma_\alpha} V_\alpha^\beta$, with $V_\alpha^\beta\cap V_\alpha^{\beta'}=\phi$, $\beta\neq\beta'$.

Let $x_0\in X$. Then $\exists\ \alpha\in\Lambda$ with $x_0\in U_\alpha$. Now $U_\alpha\cap X_0$ is open set in X_0 and now,

$$p^{-1}(U_\alpha\cap X_0)=p^{-1}(U_\alpha)\cap p^{-1}(X_0)$$
$$=\Big(\bigcup_{\beta\in\Gamma_\alpha} V_\alpha^\beta\Big)\cap\tilde{X}_0$$
$$=\bigcup_{\beta\in\Gamma_\alpha}\big(V_\alpha^\beta\cap\tilde{X}_0\big)$$

Now, since V_α^β is open in $\tilde{X}$, then $V_\alpha^\beta\cap\tilde{X}_0$ is open in $\tilde{X}_0$. Also $(V_\alpha^\beta\cap\tilde{X}_0)\cap(V_\alpha^{\beta'}\cap\tilde{X}_0)=(V_\alpha^\beta\cap V_\alpha^{\beta'})\cap\tilde{X}_0$
$=\phi\cap\tilde{X}_0=\phi,\ \beta\neq\beta'$.

As, $p:V_\alpha^\beta\to U_\alpha$ is a homeomorphism, then
$p_0:V_\alpha^\beta\cap\tilde{X}_0\longrightarrow U_\alpha\cap X_0$ is a homeomorphism.

And since $\{U_\alpha\cap X_0\}_{\alpha\in\Lambda}$ is an open cover of X_0 we conclude that

$p_0:\tilde{X}_0\longrightarrow X_0$ is a ~~coverg~~ covering map.

THEOREM

Let $p:\tilde{X}\to X$, $p':\tilde{X}'\to X'$ be covering maps. Then $p\times p':\tilde{X}\times\tilde{X}'\longrightarrow X\times X'$ is a covering map.

proof:-

$p: \tilde{X} \to X$ is surjective and $p': \tilde{X}' \to X'$ is surjective.

$\therefore$ $p \times p': \tilde{X} \times \tilde{X}' \to X \times X'$ defined,

$(p \times p')(\tilde{x}, \tilde{x}') = (x, x')$, where $\tilde{x} \in \tilde{X}$, $\tilde{x}' \in \tilde{X}'$
$x \in X$, $x' \in X'$.

Since, p & p' ~~so~~ are surjective, $\exists$ $\tilde{x} \in \tilde{X}$, $\tilde{x}' \in \tilde{X}'$ s.t.

$p(\tilde{x}) = x$ and $p'(\tilde{x}') = x'$

Now, $(\tilde{x}, \tilde{x}') \in \tilde{X} \times \tilde{X}'$ and $(p \times p')(\tilde{x}, \tilde{x}') = (x, x') \in X \times X'$

$\therefore$ $p \times p'$ is surjective

So, clearly, the map $p \times p'$ is continuous (component wise) and also surjective.

Now, $(x, x') \in X \times X'$. Then $x \in X$ and $x' \in X'$. Then there exists $U \in \{U_\alpha\}_{\alpha \in \Lambda}$ with $x \in U$ and $U' \in \{U'_\mu\}_{\mu \in \Lambda'}$ with $x' \in U'$. Here $\{U_\alpha\}_{\alpha \in \Lambda}$ and $\{U'_\mu\}_{\mu \in \Lambda'}$ are open covers of X and X' respectively satisfying covering map conditions.

Now, $p^{-1}(U) = \bigcup_{\beta \in \Gamma} V^\beta$, where each V^β is open with $V^\beta \cap V^{\beta'} = \phi$, if $\beta \neq \beta' \in \Gamma$ and $p: V_\beta \to U$ is a homeomorphism for each $\beta \in \Gamma$.

Also, $p'^{-1}(U') = \bigcup_{\gamma \in \Gamma'} V'^\gamma$, where each V'^γ is open with $V'^\gamma \cap V'^{\gamma'} = \phi$, if $\gamma \neq \gamma' \in \Gamma'$ and $p': V'^\gamma \to U'$ is a ~~homeomorp~~/ homeomorphism for each $\gamma \in \Gamma'$.

Now, $(x, x') \in U \times U'$ and $U \times U'$ is open in $X \times X'$.

$(p \times p')^{-1}(U \times U') = \bigcup_{\substack{\beta \in \Gamma \\ \gamma \in \Gamma'}} (V^\beta \times V'^\gamma)$ where $V^\beta \times V'^\gamma$ is open in $\tilde{X} \times \tilde{X}'$. Also, $V^\beta \times V'^\gamma$ is a disjoint family.

Now, it is easy to see that $p \times p': V^\beta \times V'^\gamma \to U \times U'$ is a homeomorphism as $p: V^\beta \to U$ and $p': V'^\gamma \to U'$ are homeomorphism. $\{U_\alpha \times U'_\mu\}_{\substack{\alpha\in\Lambda \\ \mu\in\Lambda'}}$ is an open cover of $X\times X'$. We conclude that $p\times p' : \tilde{X}\times\tilde{X}' \to X\times X'$ is a covering map.

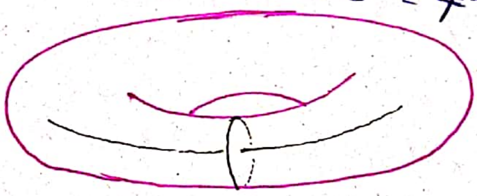
Ex $p: \mathbb{R}\to S'$ given by

$p(t) = (\cos 2\pi t, \sin 2\pi t)$ is a covering map.

$\therefore p\times p' : \mathbb{R}\times\mathbb{R} \to S'\times S'$ is a covering map, defined by

$(p\times p')(t,s) = ((\cos 2\pi t, \sin 2\pi t), (\cos 2\pi s, \sin 2\pi s))$.

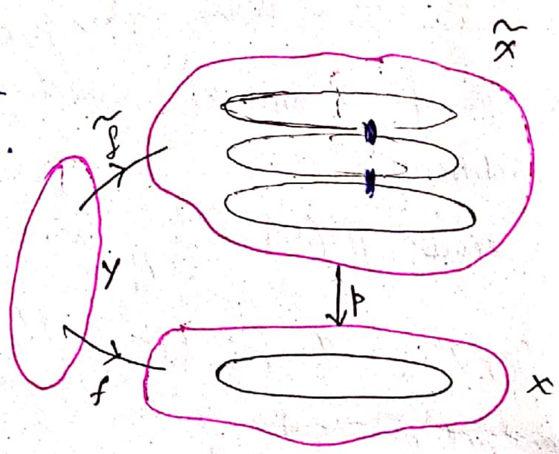
$S'\times S' = T^2$ (Torus)



Path Lifting

Definition

Let $p: \tilde{X}\to X$ be a map. Let $f: Y\to X$ be a continuous map.
A lifting of f is a map $\tilde{f}: Y\to\tilde{X}$ s.t. $p\circ\tilde{f} = f$.
$(p\circ\tilde{f})(y) = f(y)$



If f is path, then it is called path lifting.

If $\tilde{f}$ is continuous, then ~~con~~ ~~si~~ connected slice
If f is discontinuous, then disjoint slice.

Ex $p: \mathbb{R}\to S'$

$s \longmapsto (\cos 2\pi s, \sin 2\pi s)$

$f: I \to S'$

$s \longmapsto (\cos\pi s, \sin\pi s)$

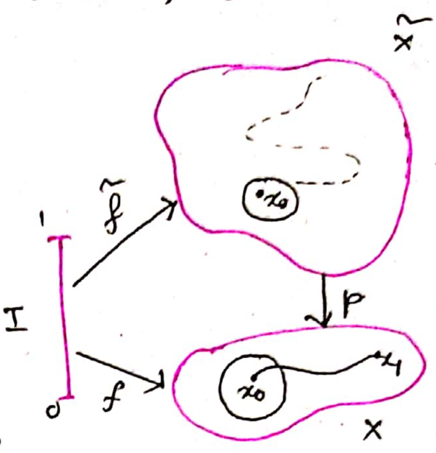
$\tilde{f}: I\to\mathbb{R}$

$\tilde{f}(s) = s/2$.

Path Lifting Property

Let $p: \tilde{X} \to X$ be a covering map with $p(\tilde{x}_0) = x_0$. If $f: I \to X$ is a path with $f(0) = x_0$, then there exists lifting of f to a path $\tilde{f}: I \to \tilde{X}$ with $\tilde{f}(0) = \tilde{x}_0$.

~~Let $p: \tilde{X} \to$~~



Let us consider an open cover $\{U_\alpha\}_{\alpha \in \Lambda}$ of X (as in the def^n of ~~cov~~ covering map).

Let $t \in (0,1)$ and $f(t) \in X$. Then $f(t) \in U_\alpha$ for some $\alpha \in \Lambda$. As $f: I \to X$ is continuous, there exists $(a_t, b_t) \subseteq (0,1)$ with $t \in (a_t, b_t)$ s.t. $f[(a_t, b_t)] \subseteq U_\alpha$.

Analogously, for $t = 0$, there exists $[0, b_0) \subseteq [0,1)$ s.t. $f[(0, b_0)] \subseteq U_\alpha$, and if $t = 1$, there exist $(a_0, 1] \subseteq (0,1]$ s.t. $f([a_0, 1]) \subseteq U_\alpha$.

Since I is compact and $\{(a_t, b_t) : t \in (0,1)\} \cup \{[0, b_0), (a_0, 1]\}$ is an open cover of I. We can choose a finite subcover, say $\{[0, b_0), (a_0, 1], (a_1, b_1), \ldots, (a_n, b_n)\}$

Let us consider $\{(a_i, b_i)\}_{i \in \{0, 1, \ldots, m\}}$

Let (possibly) us rename the elements as follows:

$0 < t_1 \leq t_2 \leq \cdots \leq t_{2m+2} < 1$. This gives a sub division $0 = s_0 < s_1 < \cdots < s_m \neq 1$ of I the property that $\forall i \in \{0, 1, \ldots, n-1\}$, $f([s_i, s_{i+1}]) \subseteq U_\alpha$ for some $\alpha \in \Lambda$

Now, let us define $\tilde{f}: I \to \tilde{X}$ followingly $\tilde{f}(0) = \tilde{x}_0$, Suppose that $\tilde{f}$ is defined on $[0, s_i]$. Then define $\tilde{f}$ in $[s_i, s_{i+1}]$ in such a way that $f([s_i, s_{i+1}]) \subseteq U_\alpha$ for some $U \in \{U_\alpha\}_{\alpha \in \Lambda}$

Let $p^{-1}(U) = \bigcup_{\beta \in \Gamma} V_\beta$. Now $(p \circ \tilde{f})(s_i) = f(s_i) \in U$.

$\Rightarrow \tilde{f}(s_i) \in p^{-1}(U) = \bigcup_{\beta \in \Gamma} V_\beta$.

$\Rightarrow \tilde{f}(s_i) \in V$ for some $V \in \{V_\beta\}_{\beta \in \Gamma}$

Define $\tilde{f}(s) = p|_V^{-1}(f(s))$: $s \in [s_i, s_{i+1}]$.

f is continuous on $[s_i, s_{i+1}]$, since $p|_V$ is a homeomorphism. $\tilde{f} : [0,1] \to \tilde{X}$, defined above is a continuous map with $\tilde{f}(0) = \tilde{x}_0$.

Now, we show that lifting is unique.

Suppose that there is another lifting $\tilde{\tilde{f}} : I \to \hat{X}$ with $\tilde{\tilde{f}}(0) = \tilde{x}_0$.

Then $\tilde{\tilde{f}}(0) = \tilde{f}(0) = x_0$.

Suppose that $\tilde{\tilde{f}}(s) = \tilde{f}(s)$ in $[0, s_i]$.

$\therefore p \circ \tilde{\tilde{f}}([s_i, s_{i+1}]) = p \circ \tilde{f}([s_i, s_{i+1}]) \subseteq U_\alpha$.

for some $\alpha \in \Lambda \Rightarrow \tilde{\tilde{f}}([s_i, s_{i+1}]) \subseteq p^{-1}(U_\alpha) = \dot{\bigcup}_{\beta \in \Gamma} V_\beta$

Since $\tilde{\tilde{f}}([s_i, s_{i+1}])$ is connected and $\tilde{\tilde{f}}(s_i) = \tilde{f}(s_i) \in V$ we have $\tilde{\tilde{f}}([s_i, s_{i+1}]) \subseteq V$.

So, for $s \in [s_i, s_{i+1}]$, $p \circ \tilde{\tilde{f}}(s) = f(s)$

$\Rightarrow \tilde{\tilde{f}}(s) \in p^{-1} f(s)$.

Also, $\tilde{\tilde{f}}(s_i) \in V$, so we must have $\tilde{\tilde{f}}(s) = p|_V^{-1}(f(s))$

$= \tilde{f}(s)$.

Thus $\tilde{\tilde{f}}(s) = \tilde{f}(s)$, $\forall s \in [s_i, s_{i+1}]$

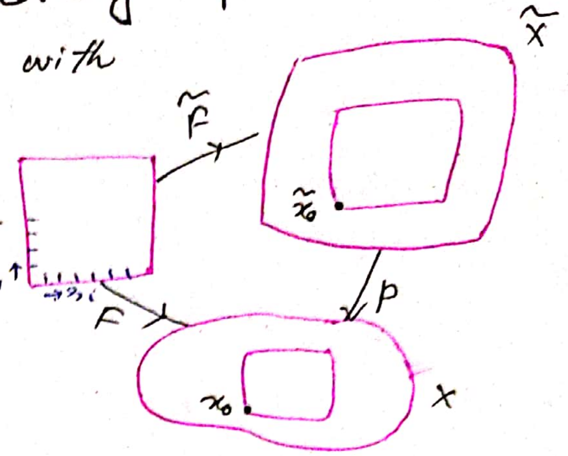
Hence proved.

Homotopy Lifting

Let $p: \tilde{X} \to X$ be a covering map with $p(\tilde{x}_0) = x_0$.

Let $F: I \times I \to X$ be continuous with $F(0,0) = x_0$. There exists lifting of F to a continuous map $\tilde{F}: I \times I \to \tilde{X}$ with $\tilde{F}(0,0) = \tilde{x}_0$.

If F is a path homotopy, then $\tilde{F}$ is also a path homotopy.



proof :-

Define $\tilde{F}(0,0) = \tilde{x}_0$

Extend $\tilde{F}$ to $I \times \{0\}$ and $\{0\} \times I$ using the technique of the last result. Let us extend $\tilde{F}$ to $I \times I$ as follows:

Let us choose a subdivision $s_0 < s_1 < s_2 < \cdots < s_m$, $t_0 < t_1 < \cdots < t_n$ of I with the property that for each rectangle $I_i \times J_j = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$, $F(I_i \times J_j) \subseteq U_\alpha$ for som $\alpha \in \Lambda$

Define $\tilde{F}$ first in $I_1 \times J_1$, then $I_2 \times J_1, \ldots, I_m \times J_1$ then $I_1 \times J_2, I_2 \times J_2, \ldots\ldots$, and so on.

Now consider $I_{i_0} \times J_{j_0}$ and suppose that $\tilde{F}$ is defined on the union A of rectangles $I_i \times J_j$ with $j < j_0$ or $j = j_0$ and $i < i_0$.

Consider $C = A \cap (I_{i_0} \times J_{j_0})$, choose $U \in \{U_\alpha\}_{\alpha \in \Lambda}$ with $F(I_{i_0} \times J_{j_0}) \subseteq U$.

Let $p^{-1}(U) = \bigcup_{\beta \in \Lambda} V_\beta$. $\tilde{F}$ is defined in C and C is Connected, i.e. $\exists$ a V_0 s.t. $\tilde{F}(C) \subseteq V_0$.

If $p_0 = p|_{V_0} : V_0 \to U$, then $(p_0 \circ \tilde{F})(x) = p \circ \tilde{F}(x) = F(x)$

$\Rightarrow \tilde{F}(x) = p_0^{-1}(F(x))$

Define $\tilde{F}(x) = p_0^{-1}(F(x))$, $\forall x \in I_{i_0} \times J_{j_0}$

Then $\tilde{F}$ is continuous according to the pasting lemma. This is infact an unique ways of defining $\tilde{F}$

Now, if F is a path homotopy, then $p(\{0\}\times I) = x_0$.

Also, $\tilde{F}(\{0\}\times I) \subseteq p^{-1}(\{x_0\})$.

Now, $p^{-1}(\{x_0\})$ has the discreate topology as a subspace of $\tilde{X}$. And also $\tilde{F}(\{0\}\times I)$ is connected. Thus $\tilde{F}(\{0\}\times I) = \{\tilde{x}_0\}$.

Similarly $\tilde{F}(\{1\}\times I)$ is a single tone set.

Thus $\tilde{F}$ is a path homotopy.

Homotopy Lifting Property

Let $p: \tilde{X} \to X$ be a covering map with $p(\tilde{x}_0) = x_0$. Consider the paths f, g in X from x_0 to x_1. Let $\tilde{f}, \tilde{g}$ be the lifting of f & g respectively in $\tilde{X}$ with $\tilde{f}(0) = \tilde{g}(0) = x_0$. If $f \simeq g$, then $\tilde{f} \simeq \tilde{g}$.

proof:-

~~Let $p: \tilde{X} \to X$ be a covering map with $p(\tilde{x}_0) = x_0$~~

Let us consider F, the path homotopy between f & g. Then $F((0,0)) = x_0$. Let $\tilde{F}: I\times I \to \tilde{X}$ be the lifting of F in $\tilde{X}$ with $\tilde{F}((0,0)) = \tilde{x}_0$.

Then $\tilde{F}(\{0\}\times I) = \{\tilde{x}_0\}$ and $\tilde{F}(\{1\}\times I) = \{\tilde{x}_1\}$.

Then $\tilde{F}(I\times\{0\})$ is a path in $\tilde{X}$ starting at $\tilde{x}_0$ i.e. a lifting of $F(I\times\{0\})$, i.e. $\tilde{F}(I\times\{0\}) = \tilde{f}$ (from uniqueness of path lifting).

Similarly, $\tilde{F}(I\times\{1\}) = \tilde{g}$. So, $\tilde{f}(1) = \tilde{g}(1) = \{x_1\}$.

and hence $\tilde{F}: I\times I \to \tilde{X}$ is a path homotopy from $\tilde{f}$ to $\tilde{g}$.

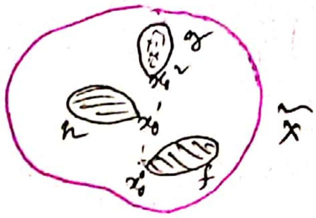
Definition

Let $p: \tilde{X} \to X$ be a covering map with $p(\tilde{x}_0) = x_0$. Given a loop f in X based at x_0, let $\tilde{f}$ be the lifting of f in $\tilde{X}$ with $\tilde{f}(0) = \tilde{x}_0$.

Define,

$$\phi : \pi_1(X, x_0) \to p^{-1}(\{x_0\})$$

$$\phi : ([f]) = \tilde{f}(1).$$



ϕ is called the lifting correspondence derive from the map p.

* Lifting correspondence ϕ is well defined:

proof:- Let $[f] = [f']$. Then the paths f and f' are homotopic. We can lift this homotopy between $\tilde{f}$ and $\tilde{f}'$. This ~~def~~ implies $\tilde{f}(1) = \tilde{f}'(1)$.

THEOREM

If $p : \tilde{X} \to X$ be a covering map with $p(\tilde{x}_0) = x_0$. If $\tilde{X}$ is a path connected, then ϕ is surjective and if $\tilde{X}$ is simply connected, then ϕ is bijective.

proof:-

Let $\tilde{x}_1 \in p^{-1}(\{x_0\})$. Since $\tilde{X}$ is path connected, there is a path $\tilde{f} : I \to \tilde{X}$ with $\tilde{f}(0) = \tilde{x}_0$ and $\tilde{f}(1) = x_1$. Then $f = p \circ \tilde{f}$ is a loop in X based at x_0 because $f(0) = p(\tilde{x}_0) = x_0$ and $f(1) = p(\tilde{x}_1) = x_0$. Then $\phi([f]) = \tilde{f}(1) = x_1$.

Let us now suppose that $\tilde{X}$ is simply connected. Since $\tilde{X}$ is simply connected it is path connected, so ϕ is already surjective. Therefore we need only check that ϕ is injective. Let us consider $[f], [g] \in \pi_1(X, x_0)$ with

$$\phi([f]) = \phi([g]).$$

Then $\tilde{f}(1) = \tilde{g}(1)$, where $\tilde{f}, \tilde{g}$ are lifting of f, g respectively with $\tilde{f}(0) = \tilde{g}(0) = \tilde{x}_0$. Since $\tilde{X}$ is simply connected then we have $[\tilde{f}] = [\tilde{g}]$, i.e. there is a path homotopy $\tilde{F}$ between $\tilde{f}$ & $\tilde{g}$. Then $F = p \circ \tilde{F}$ is a path homotopy between f & g. This implies $[f] = [g]$.

Hence ϕ is injective. And as a result ϕ is bijective.

THE FUNDAMENTAL GROUP OF CIRCLE

THEOREM

$\pi_1(S^1) \cong (\mathbb{Z}, +)$.

proof :-

S^1 is path connected. Let $p: \mathbb{R} \to S^1$ be given by $p(t) = (\cos 2\pi t, \sin 2\pi t)$. Denote $p(0) = (1,0) = x_0$.
Then $p^{-1}(\{x_0\}) = \mathbb{Z}$. Now define $\phi: \pi_1(S^1, (1,0)) \to \mathbb{Z}$ by $\phi([f]) = \tilde{f}(1)$

Since $\mathbb{R}$ is simply connected, ϕ is bijective

Let us now ~~p~~ prove that ϕ is a homomorphism. Now,

$\phi([f]\cdot[g]) = \phi([f\cdot g]) = \widetilde{f\cdot g}(1)$.

We need to show that $\widetilde{f\cdot g}(1) = \tilde{f}(1) + \tilde{g}(1)$.

Now, $\tilde{f}$ is the lift of f with $\tilde{f}(0) = 0 \in \mathbb{R}$ and $\tilde{g}$ is the lift of g with $\tilde{g}(0) = 0 \in \mathbb{R}$ and $\widetilde{f\cdot g}$ is ~~the~~ the lift of $f\cdot g$ with $\widetilde{f\cdot g}(0) = 0$.

Let $\tilde{g}': I \to \mathbb{R}$ be a path such that

$\tilde{g}'(s) = \tilde{g}(s) + \tilde{f}(1)$.

Then $(p\circ\tilde{g}')(s) = p(\tilde{g}'(s))$

$= p(\tilde{g}(s) + \tilde{f}(1))$

$= p(\tilde{g}(s)) = (p\circ\tilde{g})(s) = g(s)$.

This implies, $\tilde{g}'$ is a lift of g with $\tilde{g}'(0) = \tilde{f}(1)$.

Furthermore, $(p(\tilde{f}\cdot\tilde{g}))(s) = (f\cdot g)(s)$. Thus implies $\tilde{f}\cdot\tilde{g}$ is a lift of ~~p~~ $f\cdot g$ with $(\tilde{f}\cdot\tilde{g})(0) = 0$.

Hence $(\tilde{f}\cdot\tilde{g}')(1) = 0 + \tilde{f}(1) + \tilde{g}(1)$

$\Rightarrow (\tilde{f}\cdot\tilde{g})(1) = \tilde{f}(1) + \tilde{g}(1)$

Hence, $\phi([f]\cdot[g]) = \phi([f\cdot g]) = \widetilde{f\cdot g}(1)$

$= \tilde{f}(1) + \tilde{g}(1) = \phi([f]) + \phi([g])$.

$\therefore \phi$ is homomorphism and also bijective. Then ϕ is isomorphism $\phi_1(S^1)$ is isomorphic with S^1.

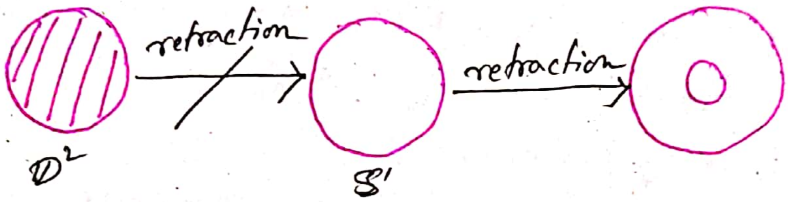
Retraction

A retraction of a space X onto $A \subseteq X$ is a continuous map $r: X \to A$ s.t. $r(a) = a$, $\forall a \in A$.

If such a map exist A is called a retract of X.

$D^2 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, $S^1 = \{(x,y) \in \mathbb{R}^2, x^2+y^2 = 1\}$

no retraction



Proposition

There is no retraction of D^2 onto S^1.

proof :-

Suppose that S^1 is a retract of D^2. Let $i: S^1 \to D^2$ be the inclusion. Then $i_*: \pi_1(S^1) \to \pi_1(D^2)$ would be injective. But $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(D^2)$ is trivial as D^2 is a convex subset of $\mathbb{R}^2$. This is a contradiction.

And ~~three~~ there is no retraction of D^2 onto S^1

Lemma

Let $h: S^1 \to X$ be a continuous map. Then the following are equivalent:

1. h is null homotopy
2. There exist a continuous map $\bar{h}: D^2 \to X$ with $\bar{h}|_{S^1} = h$.
3. h_* is trivial homomorphism of fundamental groups.

The Fundamental Theorem of Algebra

A polynomial equation

$$a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 = 0$$

of degree $n > 0$ with real or complex co-efficients has at least one (real or complex) root.

proof:-

Step I: Let $a_n x^n + \dots + a_1 x + a_0 \in \mathbb{C}[x]$, $n > 0$. Let us assume that $a_n = 1$. $\left[a_n\left(x^n + \frac{a_{n-1}}{a_n}x^{n-1} + \dots + \frac{a_1}{a_n}x + \frac{a_0}{a_n}\right); a_n \neq 0\right]$ (we can do this without loss of generality].

Indeed let us choose $c \in \mathbb{R}^+$ and set $x = cy$. This gives

$$c^n y^n + \dots + a_1 cy + a_0 = c^n\left(y^n + \frac{a_{n-1}}{c}y^{n-1} + \dots + \frac{a_0}{c^n}\right)$$

Let us choose c sufficiently large so that

$$\left|\frac{a_{n-1}}{c}\right| + \dots + \left|\frac{a_0}{c^n}\right| < 1$$

Now, if y_0 is a root of $y^n + \frac{a_{n-1}}{c}y^{n-1} + \dots + \frac{a_0}{c^n}$, then cy_0 is a root of $x^n + a_{n-1}x^{n-1} + \dots + a_0$.

Step II:

Let us consider the map $f : S^1 \to S^1$ given by $f(z) = z^n$, where z is a complex number. We show that the induced homomorphism f_* of fundamental group is injective.

Let p_0 be the standard loop in S^1,

Let $p_0 : I \to S^1$

$$p_0(s) = e^{2\pi i s} = (\cos 2\pi s, \sin 2\pi s).$$

Its image under f_* is the loop, $f_*([p_0]) = [f \circ p]$, where

$$f(p_0(s)) = (e^{2\pi i s})^n = (\cos 2\pi ns, \sin 2\pi ns).$$

This loop lifts $s \mapsto ns$ in the covering space $\mathbb{R}$. Therefore the loop $f \circ p_0$ corresponds to the integer n under the standard isomorphism of $\pi_1(S^1, b_0)$ with the integers, whereas p_0 corresponds to the number 1. Thus f_* is "multiplication by n" in the fundamental group of S^1, so that in particular, f_* is injective.

Step ~~@~~ III :

We show that if $g : S^1 \to \mathbb{R}^2 \setminus \{(0,0)\}$ is the map $g(z) = z^n$, then g is not null homotopic.

The map g equals the map f of step II followed by the inclusion map $j : S^1 \to \mathbb{R}^2 \setminus \{(0,0)\}$. Now, f_* is injective. and j_* is injective because S^1 is a retract of $\mathbb{R}^2 \setminus \{(0,0)\}$. Therefore, $g_* = j_* \circ f_*$ is injective. Thus g cannot be null homotopic. ———— (a)

Step IV

Suppose that our polynomial has not an root on D^2. Now, we consider $\bar{h} : D^2 \to \mathbb{R}^2 \setminus \{(0,0)\}$ define by

$$\bar{h}(z) = z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0.$$

Now, ~~let~~ $\bar{h}|_{S^1}$ is null homotopic, as $\bar{h}|_{S^1}$ extends to a map ~~of~~ of the unit ball into $\mathbb{R}^2 \setminus \{(0,0)\}$, ~~the map $\bar{h}|_{S^1}$ is~~ ———— (b)

~~null ho~~

Furthermore, we define a homotopy, $F : S^1 \times I \to \mathbb{R}^2 \setminus \{(0,0)\}$ where $F(z,t) = z^n + t(a_{n-1} z^{n-1} + \dots + a_1 z + a_0)$ is a homotopy between $\bar{h}|_{S^1}$ and g. ———— (c)

$$\begin{aligned}
\text{Now, } |F(z,t)| &= |z^n + t(a_{n-1} z^{n-1} + \dots + a_1 z + a_0)| \\
&\geq |z|^n - |t|\,|a_{n-1} z^{n-1} + \dots + a_0| \\
&\geq 1 - |t| \sum_{i=0}^{n-1} |a_i| > 0.
\end{aligned}$$

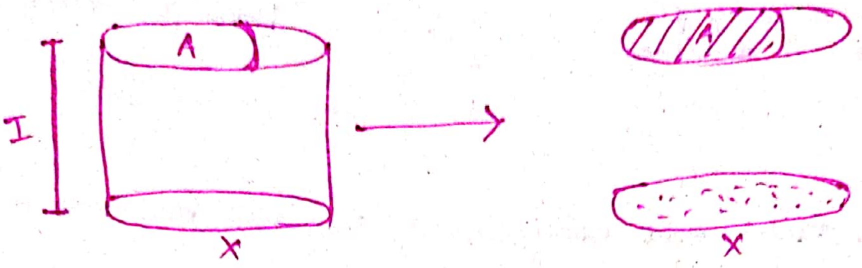
Then (a), (b), (c) are ~~Condi~~ Contradictory. So, our polynomial has a root in D^2.

Hence the theorem.

Deformation Retract and Homotopy Type

Definition

Let $A \subseteq X$. A is said to be a deformation retract of X if there is a continuous map $H: X \times I \longrightarrow X$ with $H(x,0) = x$, $\forall x \in X$ and $H(x,1) \in A$ $\forall x \in X$, $H(a,t) = a$, $\forall a \in A$.



The Homotopy H is called the deformation retraction of X onto A.

Remark

1. $r: X \to A$ s.t. $r(x) = H(x,1)$ is a retraction of X onto A.

2. H is a homotopy between I_X and $I\circ r$ where I is the inclusion map $I: A \hookrightarrow X$.

3. $A = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \neq (0,0),\ z = 0\}$

   $X = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \neq (0,0)\}$

   A is deformation retract of X.

Define $H: X \times I \to X$ by

$$H(\hat{x},t) = H(x,y,z,t) = (x,y,(1-t)z).$$

i) H is continuous

ii) $H(x,y,z,0) = (x,y,z)$, $\forall (x,y,z) \in X$

iii) $H(x,y,z,1) = (x,y,0) \in A$, $\forall (x,y,z) \in X$.

THEOREM

Let A be a deformation retract of X, $x_0 \in A$. Let $\mathcal{J}: (A, x_0) \longrightarrow (X, x_0)$ be the inclusion map. Then $\mathcal{J}_*: \pi_1(A, x_0) \to \pi_1(X, x_0)$ is an isomorphism.

proof :-

Let $r: (X, x_0) \longrightarrow (A, x_0)$, where $r(x) = H(x, 1)$ be the retraction of (X, x_0) onto (A, x_0). Then ~~$r \circ \mathcal{J}: A \to$~~ $r \circ \mathcal{J}: (A, x_0) \longrightarrow (A, x_0)$ is the identity map on I_A. ——— ①

and $(r \circ \mathcal{J})_* = (I_A)_* \Rightarrow r_* \circ \mathcal{J}_* = I_{\pi_1(A, x_0)}$

Furthermore, $\mathcal{J}_* \circ r_* = (\mathcal{J} \circ r)_*$ and $(\mathcal{J} \circ r): (X, x_0) \to (X, x_0)$ and there is a homotopy H between I_X and $\mathcal{J} \circ r$ and $H(x_0, t) = x_0$, $\forall t \in I$.

Let $f: I \to X$ be a loop with $f(0) = f(1) = x_0$. Then $H \circ (f \times I_I): I \times I \to X$ is a path homotopy between $I_X \circ f$ and ~~$(\mathcal{J} \circ f)$~~ $(\mathcal{J} \circ r) \circ f$ as $H \circ (f \times I_I)(0, t) = H(x_0, t) = x_0$, $\forall t \in I$. and $H \circ (f \times I_I)(1, t) = H(x_0, t) = x_0$, $\forall t \in I$.

So, $[I_X \circ f] = [(\mathcal{J} \circ r) \circ f]$

$$\Rightarrow I_*(f) = (\mathcal{J} \circ r)_*(f)$$

$$\Rightarrow I_{\pi_1(X, x_0)} = (\mathcal{J} \circ r)_* = \mathcal{J}_* \circ r_*$$

$$\Rightarrow \mathcal{J}_* \circ r_* = I_{\pi_1(X, x_0)}$$

$$r_* \circ \mathcal{J}_* = I_{\pi_1(A, x_0)}.$$

i.e. $\mathcal{J}_*$ is invertible and $\mathcal{J}_*$ is bijective.

Now, we show that $\mathcal{J}_*$ is a homeomorphism.

$$\mathcal{J}_*([f \cdot g]) = [\mathcal{J} \circ (f \cdot g)]$$
$$= (\mathcal{J} \circ f) \cdot (\mathcal{J} \circ g)$$
$$= \mathcal{J}_*[f] \cdot \mathcal{J}_*[g].$$

$\therefore \mathcal{J}_*: \pi_1(A, x_0) \longrightarrow \pi_1(X, x_0)$ isomorphism.

Ex

$g: S^2 \hookrightarrow \mathbb{R}^3 \setminus \{(0,0,0)\}$. Here S^2 is a retraction of $\mathbb{R}^3 \setminus \{(0,0,0)\}$. $g_*: \pi_1(S^2, x_0) \to \pi_1(\mathbb{R}^3 \setminus \{(0,0,0)\}, x_0)$

~~Since ... are simply connected, as~~

S^2 is simply connected as $\mathbb{R}^3 \setminus \{(0,0,0)\}$ are simply connected.

Real Projection Plane $\mathbb{R}P^2$

$\mathbb{R}P^2$ = set of all one-dimensional subspaces of $\mathbb{R}^3$
= set of all lines passing through the origin

$\mathbb{R}P^2 = S^2/_{\sim}$, where $x \sim -x$, $x \in S^2$

$= \{[x] : x \in S^1\}$

$= \{[x] : x \in S^1\} = \{\{\pm x\} : x \in S^2\}$

Define, $p: S^2 \to \mathbb{R}P^2$, by $p(x) = [x]$.

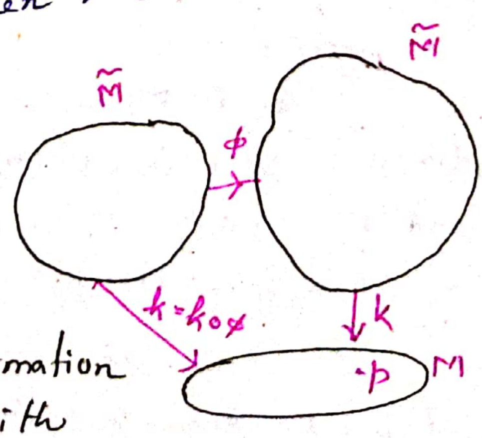
p is a two sheeted covering map from S^2 to $\mathbb{R}P^2$. Since S^2 is simply connected, then $\pi_1(S^2)$ contains just one element.

Therefore, the fundamental group $\pi_1(\mathbb{R}P^2)$ contains two elements.

Then $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2 \cong \mathbb{Z}/2\mathbb{Z}$.

Deck Transformation

Let $k: \tilde{M} \to M$ be a covering map and a diffeomorphism $\phi: \tilde{M} \to \tilde{M}$ be such that $k \circ \phi = k$. Then ϕ is said to be a deck transformation of the the covering map k.



* The ϕ merely rearranges the points in each fiber $k^{-1}(p)$, where $p \in M$

The set $\mathcal{D}$ of all deck transformation of a covering map forms a group with composition function as the group operation.

Ex

Let us consider the covering map $exp: \mathbb{R} \to S'$ given by $exp(t) = (\cos t, \sin t)$.

Now, $exp(t + 2\pi n) = (\cos t, \sin t) = exp(t)$.

Thus any deck transformation must carry each $t \in \mathbb{R}$ to a point $(t + 2\pi n)$, and by continuity the integer n is independent of t. Hence the deck transformation group $\mathcal{D}$ of this covering consists of all translations $\phi_n(t) \equiv t + 2n\pi$.

* The function $n \mapsto \phi_n$ is an isomorphism from the additive group of integers $\mathbb{Z}$ to $\mathcal{D}$. So, $\mathcal{D}$ is infinite cyclic

* A deck transformation ϕ of $k: \tilde{M} \to M$ is basically a lift of k through k.

* A deck transformation of a connected covering is determine by its unique value at a single point. Explicitly, if ϕ & $\psi \in \mathcal{D}$ and $\phi(p) = \psi(p)$ for some $p \in \tilde{M}$, then $\phi = \psi$. In particular, since the identity map of $\tilde{M}$ is a deck transformation on the existance of a single fixed point, $\phi(p) = p$, implies that ϕ is the identity map.

Ex $\psi(p) = \phi(p)$ for some $p \in \tilde{M}$

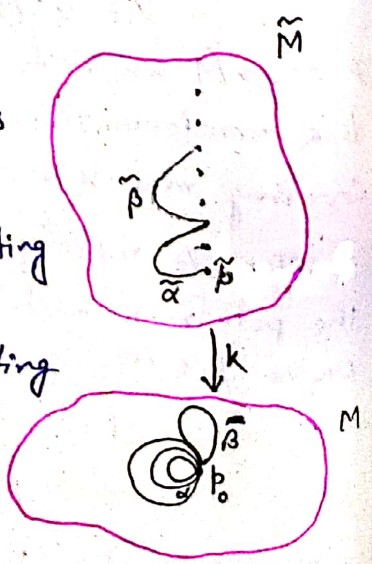
$\Rightarrow \psi(p) = \phi(p)$, $\forall p \in \tilde{M}$.

THEOREM

If $k: \tilde{M} \to \tilde{M}$ is a simply connected covering. Then its deck transformation group $\mathcal{D}$ is isomorphic to the fundamental group $\pi_1(M)$ of M.

proof :-

Let $\pi_1(M)$ be based at $p_0 \in M$. Let us fixed a point $\tilde{p} \in k^{-1}(p_0)$. Let $a \in \pi_1(M)$ and let $\alpha: [0,1] \to M$ be the loop ~~reaf~~ representing a, i.e. $\alpha(0) = \alpha(1) = p_0$ and $[\alpha] = a$.



Let $\tilde{\alpha}$ be the unique lift of α starting at $\tilde{p}$. We know that the point $\tilde{\alpha}(1)$ depends only on the choice of $\tilde{p}$ and $a = [\alpha]$.

Let $\phi_a : \tilde{M} \to \tilde{M}$ be the unique deck transformation s.t. $\phi_a(\tilde{p}) = \tilde{\alpha}(1)$. The ~~assignme~~ assignment $a \mapsto \phi_a$ gives a map $\pi_1(M) \mapsto \mathcal{D}$.

For $a = [\alpha]$ and $b = [\beta]$ choosen from $\pi_1(M)$, the lift ~~of~~ $\tilde{\alpha}$ and $\tilde{\beta}$ and we see that $\phi_a \circ \tilde{\beta}$ is a path from $\phi_a(\tilde{p})$ to $\phi_a(\tilde{\beta}(1)) = \phi_a(\phi_b(\tilde{p}))$.

Thus the path $\tilde{\gamma} = \tilde{\alpha} * (\phi_a \circ \tilde{\beta})$ is defined.

Now since $k \circ \phi_a = k$, we have

$$
\begin{aligned}
k \circ \tilde{\gamma} &= k \circ (\tilde{\alpha} * (\phi_a \circ \tilde{\beta})) \\
&= (k \circ \tilde{\alpha}) * (k \circ (\phi_a \circ \tilde{\beta})) \\
&= \alpha * ((k \circ \phi_a) \circ \tilde{\beta}) \\
&= \alpha * (k \circ \tilde{\beta}) = \alpha * \beta
\end{aligned}
$$

Since $\tilde{\alpha} * \tilde{\beta}$ represents the element $ab \in \pi_1(M)$, we have $\phi_{ab}(\tilde{p}) = \tilde{\gamma}(1) = \phi_a(\phi_b(\tilde{p}))$. Since $\tilde{M}$ is ~~st~~ simply connected we must have $\phi_{ab} = \phi_a \circ \phi_b$. Thus the map $a \mapsto \phi_a$ is a group homomorphism.

Let $\psi \in \mathcal{D}$. Let us consider the curve $\tilde{\gamma}$ from $\tilde{p}$ to $\psi(\tilde{p})$. Then The we have $k \circ \tilde{\gamma}$ to be a loop based at p_0. Then $c = [k \circ \tilde{\gamma}] \in \pi_1(M)$ and $\psi = \phi_c$.

Thus the map $a \mapsto \phi_a$ is onto.

Finally, if ϕ_a is the identity map, then any loop $\alpha \in [\alpha] = a$ lifts to a loop $\tilde{\alpha}$ based at $\tilde{p}$. But $\tilde{M}$ is simply connected, and so, $\tilde{\alpha}$ is homotopic to a constant map to $\tilde{p}$ and its projection α is therefore a constant map to p. Thus $a = [\alpha] = 0$, the identity element in $\pi_1(M)$ and hence kernal of the homomorphism is $\{0\}$. Hence the homomorphism is one-one.

Hence $\pi_1(M)$ and $\mathcal{D}$ are isomorphic.

Homotopy Equivalence

Let $f: X \to Y$ and $g: Y \to X$ be continuous maps. Suppose that the map $g \circ f : X \to X$ is homotopic to the identity map to X, and the map $f \circ g : Y \to Y$ is homotopic to the identity map of Y. Then the maps f and g are called homotopy equivalences, and each is said to be a homotopy inverse of ~~a~~ the other.

* It is straight forward to show that if $f: X \to Y$ is a homotopy equivalence of X with Y and $h: Y \to Z$ is a homotopy equivalence of Y with Z, then $h \circ f : X \to Z$ is a homotopy equivalence of X with Z. It follows that the relation of homotopy equivalence is an equivalence relation. Two spaces that are homotopy equivalent are said to have the same homotopy here.

* If A is a deformation retract of X, then A has the same homotopy type as X. For let $j: A \to X$ be the inclusion mapping and let $r: X \to A$ be the retraction mapping. Then the composite $r \circ j$ equals the identity map of A, and the composite $j \circ r$ is by hypothesis homotopic to the identity map of X.

Lemma

Let $h, k : X \to Y$ be continuous maps; let $h(x_0) = y_0$ and $k(x_0) = y_1$. If h and k are homotopic, there is a path α in Y from y_0 to y_1 such that $k_* = \hat{\alpha} \circ h_*$. Indeed, if $H: X \times I \to Y$ is the homotopy between h and k, then α is the path $\alpha(t) = H(x_0, t)$.

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "h_*"] \arrow[dr, "k_*"'] & \pi_1(Y, y_0) \arrow[d, "\hat{\alpha}"] \\
& \pi_1(Y, y_1)
\end{tikzcd}$$

90 4.15

proof :- Let $f : I \to X$ be a loop in X based at x_0. We must show that $k_*[f] = \hat{\alpha}(h_*([f]))$.

This equation states that $[k \circ f] = [\bar{\alpha}] * [h \circ f] * [\alpha]$

or, $[\alpha] * [k \circ f] = [h \circ f] * [\alpha]$.

This is the equation we shall verify.

To begin, consider the loops f_0 and f_1 in the space $X \times I$ given by the equations

$$f_0(s) = (f(s), 0) \text{ and } f_1(s) = (f(s), 1).$$

Consider also the path c in $X \times I$ given by the eq^n

$$c(t) = (x_0, t)$$

Then $H \circ f_0 = h \circ f$ and $H \circ f_1 = k \circ f$, while $H \circ c$ equals ~~to~~ the path α.

Let $F : I \times I \to X \times I$ be the map $F(s,t) = (f(s), t)$. Consider the following paths in $I \times I$, which run along the four edges of $I \times I$:

$$\beta_0(s) = (s, 0) \quad \text{and} \quad \beta_1(s) = (s, 1)$$

$$\gamma_0(t) = (0, t) \quad \text{and} \quad \gamma_1(t) = (1, t).$$

Then $F \circ \beta_0 = f_0$ and $F \circ \beta_1 = f_1$ while $F \circ \gamma_0 = F \circ \gamma_1 = c$.

The broken-line paths $\beta_0 * \gamma_1$ and $\gamma_0 * \beta_1$ are paths in $I \times I$ from $(0,0)$ to $(1,1)$; since $I \times I$ is convex, there is a path homotopy G between them. Then $F \circ G$ is a path homotopy in $X \times I$ between $f_0 * c$ and $c * f_1$. And $H \circ (F \circ G)$ is a path homotopy in Y between

$$(H \circ f_0) * (H \circ c) = (h \circ f) * \alpha \text{ and}$$

$$(H \circ c) * (H \circ f_1) = \alpha * (k \circ f).$$

Hence proved.

Scanned with CamScanner

Corollary

1. Let $h, k : X \to Y$ be homotopic continuous maps; let $h(x_0) = y_0$ and $k(x_0) = y_1$. If h_* is injective, or surjective, or trivial, so is k_*.

2. Let $h : X \to Y$. If h is nulhomotopic, then h_* is the trivial homomorphism.

THEOREM

Let $f : X \to Y$ be continuous; let $f(x_0) = y_0$. If f is a homotopy equivalence, then

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0).$$

is an isomorphism.

Proof:-

Let $g : Y \to X$ be a homotopy inverse for f. Consider the maps

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (X, x_1) \xrightarrow{f} (Y, y_1).$$

where $x_1 = g(y_0)$ and $y_1 = f(x_1)$. We have the corresponding induced homomorphism:

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "(f_{x_0})_*"] & \pi_1(Y, y_0) \arrow[dl, "g_*"] \\
\pi_1(X, x_1) \arrow[r, "(f_{x_1})_*"] & \pi_1(Y, y_1)
\end{tikzcd}$$

[Here we have to distinguish between the homomorphisms induced by f relative to two different base points.]. Now

$$g \circ f : (X, x_0) \longrightarrow (X, x_1)$$

is by hypothesis homotopic to the identity map, so there is a path α in X such that

$$(g \circ f)_* = \hat{\alpha} \circ (i_X)_* = \hat{\alpha}.$$

It follows that $(g \circ f)_* = g_* \circ (f_{x_0})_*$ is an isomorphism.

Similarly, because $f \circ g$ is homotopic to the identity map i_Y, the homomorphism $(f \circ g)_* = (f_{x_1})_* \circ g_*$ is an isomorphism.

The first fact implies that g_* is surjective; and the 2nd implies that g_* is injective. Therefore g_* is an isomorphism. Applying the 1st equation once again, we conclude that

$$(f_{x_0})_* = (g_*)^{-1} \circ \hat{\alpha}$$

So, that $(f_{x_0})_*$ is also an isomorphism.

Note that although g is a homotopy inverse for f, the homomorphism g_* is not an inverse for the homomorphism $(f_{x_0})_*$.

* T^2 isomorphic to $S^1 \times S^1$.

We know, that $\pi_1(S_1)$ is isomorphic to $\mathbb{Z}$.

$$\pi_1(T^2) \approx \mathbb{Z} \times \mathbb{Z}.$$